# LAGRANGIAN NON-SQUEEZING AND A GEOMETRIC INEQUALITY

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Abstract. We prove that if the unit codisc bundle of a closed Riemannian manifold embeds symplectically into a symplectic cylinder of radius one then the length of the shortest nontrivial closed geodesic is at most half the area of the unit disc.

## 1. Introduction

Consider a metric $g$ on the unit circle $S^1 = \partial D$. After a reparametrization by arclength $g = r^2 g_0$ becomes a positive multiple of the standard metric on $S^1$. If the unit codisc bundle of $(S^1, g)$ embeds into the unit disc $D$ preserving the area and the orientation then

$$2\,\mathrm{length}_g(S^1) \leq \pi,$$

which is equivalent to $r \leq \frac{1}{4}$. In other words

$$\inf(g) \leq \frac{\pi}{2},$$

where $\inf(g)$ denotes the length of the shortest nontrivial closed geodesic. We prove that this inequality has a symplectic generalization for any closed Riemannian manifold $(L, g)$ such that its unit codisc bundle $D^*(g)L$ has a symplectic embedding into the cylinder $Z = D \times \mathbb{R}^{2n-2}$ (provided with the standard split symplectic form). In fact, in dimension $\geq 4$ it is only required for the symplectic embedding to exist in a neighbourhood of the unit cosphere bundle $S^*(g)L$, see Corollary 3.3.

For the proof of this result we introduce a capacity $\ell$, see Theorem 3.1. For a given set $U$, $\ell$ measures the largest minimal total action of null-homologous Reeb links on a contact type hypersurface in $U$ diffeomorphic to a unit cotagent bundle. This is a variant of the capacity introduced in [9, 10], cf. [18]. It fits into a larger class of the so called *embedding capacities*, see [4].

The first of these embedding capacities appeared in the unpublished work [5], cf. [4]. In [5] Cieliebak and Mohnke defined a Lagrangian embedding capacity for 2-connected symplectic manifolds $(V, \omega)$

$$c_L(V, \omega) := \sup\{\inf(L) \mid L \subset (V, \omega)\},$$

where $\inf(L)$ denotes the least positive symplectic area of a smooth disc in $V$ with boundary on $L$. Here the supremum runs over all Lagrangian *tori* in $(V, \omega)$. Its values on the unit symplectic cylinder $Z$, unit ball $B$, and unit polydisc $P$ are

$$c_L(Z) = \pi, \qquad c_L(B) = \frac{\pi}{n}, \qquad \text{and} \qquad c_L(P) = \pi.$$

2010 *Mathematics Subject Classification.* 53D35, 53C22.

This capacity gives an alternative proof of the Ekeland-Hofer non-squeezing theorem [7, Corollary 3], which states that if the polydisc

$$P(r_1,\ldots,r_n) := D_{r_1}\times\ldots\times D_{r_n},$$

with radii $0<r_1\leq\ldots\leq r_n$, embeds symplectically into the ball $B_R$ of radius $R$, then $\sqrt{n}\,r_1\leq R$. Generalizing to arbitrary Lagrangian submanifolds, one obtains, as in Swoboda and Ziltener's work [15, 16], a symplectic capacity $c_L\leq a_L$ for 2-connected symplectic manifolds $(V,\omega)$ via

$$a_L(V,\omega) := \sup\{\inf(L)\,|\,L\subset(V,\omega)\text{ closed Lagrangian submanifold}\},$$

such that

$$a_L(Z)=\pi,\qquad a_L(B)\geq\frac{\pi}{2}.$$

The precise value on the unit ball is not known. In fact Swoboda and Ziltener a capacity to a large class of coisotropic submanifolds for every possible codimension and prove non-squeezing results for so-called small sets, see [15, 16].

Theorem 3.1 can be seen as another example in this direction. Moreover, in Corollary 2.3 and 2.6 we give further non-squeezing results for Lagrangian submanifolds, using Chekanov's elementary tori, see [2], Damian's proof of the Audin conjecture, see [6], and Lagrangian embedding capacities, which we introduce in Section 2.

## 2. Measuring the area

We are interested in **special capacities** $a$ on the standard symplectic vector space $\mathbb{R}^{2n}$, which are (1) monotone on subsets of $\mathbb{R}^{2n}$, i.e. $a(U_1)\leq a(U_2)$ provided $U_1\subset U_2$, (2) invariant under *global* symplectomorphisms of $\mathbb{R}^{2n}$, (3) conformal in the sense that $a(rU)=r^2a(U)$ for all $U\subset\mathbb{R}^{2n}$ and $r\in\mathbb{R}$, and (4) satisfy

$$a(Z)<\infty,\qquad a(B)>0,$$

see [12, p. 172]. The aim is to measure the minimal symplectic area $\inf(L)$ of closed Lagrangian submanifolds $L$ in $\mathbb{R}^{2n}$ among *all* smooth discs attached to $L$. In other words we consider the Liouville class $\lambda_L=[\lambda|_{TL}]$ for any primitive $\lambda$ of $\mathrm{d}\mathbf{x}\wedge\mathrm{d}\mathbf{y}$. The image of $H_1(L;\mathbb{Z})$ under $\lambda_L$ generates a subgroup $\Lambda_L$ of $\mathbb{R}$. If this group is discret, we call $L$ **rational**, and $\inf(L)$ is the positive generator of $\Lambda_L$; otherwise $\inf(L)$ is zero, see [13].

For our first version of a special capacity we consider for real numbers

$$0<r_1\leq\ldots\leq r_n$$

the **elementary Lagrangian tori**

$$T(r_1,\ldots,r_n) := \partial D_{r_1}\times\ldots\times\partial D_{r_n}$$

in $\mathbb{R}^{2n}$. Notice, that $\inf(L)=\pi r_1^2$ if the radii $r_1,\ldots,r_n$ are rationally independent. We call two closed Lagrangian submanifolds of $\mathbb{R}^{2n}$ **symplectomorphic**, if there exists a global symplectomorphism of $\mathbb{R}^{2n}$, which maps one to the other. It follows from [2, Theorem A] that the first radius

$$r_1=r_1(L)$$

of a Lagrangian torus $L$ symplectomorphic to $T(r_1,\ldots,r_n)$ is an invariant under global symplectomorphisms.

**Theorem 2.1.** *For subsets $U$ in $\mathbb{R}^{2n}$, the quantity*

$$a_e(U) := \sup\left\{\pi\big(r_1(L)\big)^2 \mid L \subset U\right\},$$

*where the supremum is taken over all Lagrangian tori $L$ symplectomorphic to an elementary torus, defines a special capacity in $\mathbb{R}^{2n}$ such that*

$$a_e(Z) = \pi, \qquad a_e(B) = \frac{\pi}{n}.$$

*Proof.* We only have to verify the normalization axiom. For the lower bounds consider the tori $T_1$ and $T_{1/\sqrt{n}}$, which have minimal symplectic action $\pi$ and $\pi/n$. To obtain upper bounds consider a Lagrangian torus $L$ in $\mathbb{R}^{2n}$, which is symplectomorphic to the elementary torus $T(r_1, \ldots, r_n)$. For $r_1 = r_1(L)$ the values of the first and $n$-th Ekeland-Hofer capacity of $L$ are

$$c_1^{\mathrm{EH}}(L) = \pi r_1^2, \qquad c_n^{\mathrm{EH}}(L) = n\pi r_1^2,$$

see [2, Theorem 2.1]. The claim follows now from

$$c_1^{\mathrm{EH}}(Z) = \pi = c_n^{\mathrm{EH}}(B)$$

and the monotonicity property of the Ekeland-Hofer capacities, see [7]. □

**Remark 2.2.** Because $c_1^{\mathrm{EH}}$ takes the value $\pi r_1^2$ on the polydisc $P(r_1, \ldots, r_n)$ the proof shows that $a_e\big(P(r_1, \ldots, r_n)\big) = \pi r_1^2$.

As a direct consequence of the theorem, the torus $T(r_1, \ldots, r_n)$ admits a global symplectic embedding into the symplectic cylinder $Z_R$ of radius $R$ if and only if $r_1 \leq R$. This non-squeezing result follows alternatively from the stronger [3, Main Theorem], which gives an upper bound on the area of a non-constant holomorphic disc (for example for the standard complex structure) attached to $T(r_1, \ldots, r_n)$ by its displacement energy. The rational case was observed by Sikorav in [14]. Note that Sikorav's theorem implies the general case by approximating irrational radii by rational numbers.

**Corollary 2.3.** *If the torus $T(r_1, \ldots, r_n)$ admits a global symplectic embedding into the ball $B_R$ of radius $R$, then $\sqrt{n}\, r_1 \leq R$.*

**Remark 2.4.** This follows alternatively with the Cieliebak-Mohnke capacity, see [5], via an approximation by rational Lagrangian tori.

A second special capacity for $\mathbb{R}^{2n}$ can be constructed as follows: Consider closed connected monotone Lagrangian submanifolds $L \subset \mathbb{R}^{2n}$, which admit a metric of non-positive sectional curvature (and are therefore, by the Hadamard-Cartan Theorem, aspherical). Notice, that $L$ is allowed to be non-orientable, so that for example in dimension 4, the curvature condition is not a restriction, see [11, 0.4.A$_2$].

**Theorem 2.5.** *For subsets $U$ in $\mathbb{R}^{2n}$, the quantity*

$$a_m(U) := \sup\{\inf(L) \mid L \subset U\},$$

*where the supremum is taken over all Lagrangian submanifolds $L \subset \mathbb{R}^{2n}$ as described above, defines a special capacity in $\mathbb{R}^{2n}$ such that*

$$a_m(Z) = \pi, \qquad a_m(B) = \frac{\pi}{n}.$$

*Proof.* We only have to show that $a_m(Z) = \pi$ and $a_m(B) = \pi/n$. The tori $T_1$ and $T_{1/\sqrt{n}}$ yield lower bounds. Uniform upper bounds are obtained as follows: Consider a Lagrangian submanifold $L \subset \mathbb{R}^{2n}$ as above. Because $L$ is monotone the Liouville class $\lambda_L$ and the Maslov class $\mu_L$ are related by

$$\lambda_L = \eta\mu_L$$

for some $\eta > 0$. By Damian's proof of the Audin conjecture we find a closed curve $\gamma$ on $L$, such that $\mu_L(\gamma) \leq 2$ (equality if and only if $L$ is orientable), see [6, Theorem 1.5.(a)]. This gives

$$\inf(L) \leq \lambda_L(\gamma) \leq 2\eta.$$

Moreover, by Bates [1, Theorem 3], the $k$-th Ekeland-Hofer capacity satisfies

$$2k\eta \leq c_k^{\mathrm{EH}}(L),$$

where the curvature condition is used, so that

$$\inf(L) \leq \frac{c_k^{\mathrm{EH}}(L)}{k}.$$

The claim follows now from the properties of the first and $n$-th Ekeland-Hofer capacity. □

**Corollary 2.6.** *Let $L \subset B_R$ be a closed connected Lagrangian submanifold. Then*

$$\inf(L) \leq \frac{\pi}{n}R^2$$

*provided $L$ is monotone and admits a metric of non-positive sectional curvature.*

**Remark 2.7.** The case of monotone tori follows from the Cieliebak-Mohnke capacity [5].

## 3. Measuring the length

For irrational Lagrangian submanifolds the symplectic area can be arbitrary small, thus not giving a sensible invariant. But the length of closed unit speed geodesics on $L$ for certain Riemannian metrics is an alternative way to measure the size of Lagrangian submanifolds $L$ symplectically.

[9, 10] construct a capacity

$$c(V,\omega) = \sup\{\inf\nolimits_\ell(\alpha) \,|\, \exists \text{ contact type embedding } (M,\alpha) \hookrightarrow (V,\omega)\}$$

for all symplectic manifolds $(V,\omega)$ with dimension $\geq 4$. Here $\inf_\ell(\alpha)$ is the infimum of the total action of null-homologous Reeb links on the closed contact manifold $(M,\alpha)$. The supremum is taken over all embeddings $j\colon M \hookrightarrow V$, such that near $j(M)$ there is a Liouville vector field $Y$ for $\omega$ such that $\alpha = j^*(i_Y\omega)$. Notice that this is equivalent to $d\alpha = j^*\omega$ for the contact form $\alpha$, see [12, p. 119].

If one restricts, in the definition of $c$, to manifolds $M$ diffeomorphic to the unit cosphere bundle $S^*Q$ of closed Riemannian manifolds $Q$, one obtains a capacity which we denote by $\ell$.

**Theorem 3.1.** *For symplectic manifolds $(V,\omega)$ with dimension $\geq 4$, the quantity $\ell(V,\omega)$ is an intrinsic capacity such that*

$$\ell(Z) = \pi, \qquad \ell(B) \geq \frac{\pi}{n}.$$

*Proof.* Because of $\ell \leq c$ we only need to compute the values on the ball and the cylinder. Identify the cotangent bundle of the unit circle $S^1 = \partial D$ with $(\mathbb{R}\times S^1, s\mathrm{d}t)$ and consider polar coordinates on $\mathbb{C}$, such that the radial Liouville primitive of the standard symplectic form becomes $\frac{1}{2}r^2\mathrm{d}\theta$. For $a > 0$ we define a symplectic embedding

$$\varphi_a(s,t) = \sqrt{a+2s}\,\mathrm{e}^{it}$$

of $\{s > -\frac{a}{2}\}$ into $\mathbb{C}$. The image of the $b$-codisc bundle

$$D_b^*S^1 = (-b,b)\times S^1,$$

$b \in (0, \frac{a}{2})$, is the annulus

$$A(a,b) = A_{\sqrt{a+2b},\sqrt{a-2b}} = D_{\sqrt{a+2b}} \setminus \overline{D}_{\sqrt{a-2b}}.$$

For real numbers $a_1,\dots,a_n,b_1,\dots,b_n$ with $0 < b_j < \frac{a_j}{2}$ for $j = 1,\dots,n$ the embedding

$$\varphi_{a_1}\times\ldots\times\varphi_{a_n}$$

maps $D_{b_1}^*S^1\times\ldots\times D_{b_n}^*S^1$ onto the polyannulus $A(a_1,b_1)\times\ldots\times A(a_n,b_n)$ symplectically.

In order to compute the quantity $\ell$ we consider the $b$-cosphere bundle $S_b^*T^n$ of the flat torus $T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ with the canonical contact form. Each closed geodesic induces a null-homologous Reeb link with two components, corresponding to the opposing orientations of the geodesic. Hence, the smallest total action $\inf_\ell$ equals $4\pi b$, see [8, Section 1.5]. Because the $b$-codisc bundle $D_b^*T^n$ is contained in $(D_b^*S^1)^n$ the images of the $b'$-cosphere bundles under $(\varphi_a)^n$ for $b' < b$ are hypersurfaces of contact type. Taking the limits $a\downarrow\frac{1}{2}$ and $b\uparrow\frac{1}{4}$, resp., $a\downarrow\frac{1}{2n}$ and $b\uparrow\frac{1}{4n}$ proves the claim. □

**Remark 3.2.** For $\varepsilon > 0$ sufficiently small the disc of radius $2\sqrt{b}-\varepsilon$ embeds into the square $(-b,b)\times(0,2\pi)$ preserving the orientation and the area, cf. [12, p. 171]. Composing this with $\varphi_{a_1}\times\ldots\times\varphi_{a_n}$ appropriately yields an symplectic embedding of the polydisc

$$P\big(2\sqrt{b_1}-\varepsilon,\dots,2\sqrt{b_n}-\varepsilon\big)$$

into the polyannulus

$$A(a_1,b_1)\times\ldots\times A(a_n,b_n) \subset P\big(\sqrt{a_1+2b_1},\dots,\sqrt{a_n+2b_n}\big).$$

Therefore, as in the proof above, one shows that $\ell\big(P(r_1,\dots,r_n)\big) = \pi r_1^2$. Moreover, if we consider the metric on $T^n$ induced from $\mathbb{R}^n$ we see, together with the remark after [10, Theorem 4.5], that

$$\ell\big(D_{b_1}^*T^n\big) = 4\pi b_1 = \ell\big(D_{b_1}^*S^1\times\ldots\times D_{b_n}^*S^1\big),$$

where we assume that $0 < b_1 \leq \ldots \leq b_n$.

Consider a closed monotone Lagrangian submanifold $L \subset \mathbb{R}^{2n}$, which admits a metric $g$ of non-positive sectional curvature. By Weinstein's neighbourhood theorem [17] there exists $r > 0$ such that the $r$-codisc bundle of $L$ embeds symplectically. We denote its image by $U_r \subset \mathbb{R}^{2n}$. Then [1, Theorem 2.1] implies that for all $k \in \mathbb{N}$

$$\inf(L) + r\inf(g) \leq c_k^{\mathrm{EH}}(U_r),$$

where $\inf(g)$ denotes the length of the shortest nontrivial closed geodesic of $(L,g)$. In particular, if $U_r \subset Z_R$ then $r\inf(g) \leq \pi R^2$. This observation generalizes to the following non-squeezing result.

**Corollary 3.3.** *Let $(Q, g)$ be a closed Riemannian manifold. If a neighbourhood of the $r$-cosphere bundle in $T^*Q$ embeds into $Z_R$ symplectically, then*

$$2r \inf(g) \leq \pi R^2.$$

*Proof.* The claim is an application of the capacity $\ell$. Notice that $\inf_\ell(\alpha_r) = 2r \inf(g)$ if computed with respect to the restriction $\alpha_r$ of the canonical Liouville form to $TS_r^*(g)Q$. □

**Acknowledgement.** I thank Hansjörg Geiges, Janko Latschev, and Stefan Suhr for their comments on the first version of these notes, Felix Schlenk for directing my attention to the work of S. M. Bates [1] during his stay at the Universität Leipzig. Further I would like to thank Fabian Ziltener for drawing my interest towards this problem in the first place. Part of the research in this article was carried out during the conference *Periodic Orbits In Contact and Riemannian Geometry* from September 3rd through September 7th 2012 in Le Touquet-Paris-Plage. I would like to thank the organizers Juan-Carlos Álvarez Paiva and Florent Balacheff as well as Urs Fraunefelder, Emmanuel Opshtein, Yaron Ostrover, Federica Pasquotto, and Ana Rechtman for many stimulating discussions.

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